Intermediate Mathematical Olympiad

Hamilton paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. There is a water tank which is a cuboid with base area, x, and height, h. Standing in the tank are three identical metal columns, each a cylinder of height, h, and base a circle of area, y. The tank is then filled with a volume, v, of water.
When one column is removed, the water level drops by 50.
When a second column is removed, the water level drops by a further 30.
Find how much further the water level drops when the last column is removed.

SOLUTION

The first scenario gives this equation.

$$v = h(x - 3y)$$

The second scenario gives this equation.

$$v = (h - 50)(x - 2y)$$

The third scenario gives this equation.

$$v = (h - 80)(x - y)$$

Equating the first two expressions for v gives this equation.

$$50x - 100y = yh$$

Equating the first and third expressions for v gives this equation.

$$80x - 80y = 2yh$$

Substituting in for $2yh$ gives this equation.

$$100x - 200y = 80x - 80y$$

Therefore, $x = 6y$.

Substituting into the previous equation gives $480y - 80y = 2yh$.

Since $y \neq 0$, $h = 200$ and $v = 600y$.

When the last column is removed, the height of the water level is $\frac{v}{x} = \frac{600y}{6y} = 100$.

Therefore, the water drops from 120 to 100, which is a further 20.

2. There are two types of tile to tile a floor. One is a square tile 5 by 5 and the other is a rectangular tile 4 by 9. The floor is 140 long by 80 wide. It is to be completely tiled with no gaps, none of them overlapping and none going outside the edges of the floor using the minimum number of tiles possible.
Demonstrate how to do this and prove that it cannot be done with fewer tiles.

SOLUTION

Let there be x of the 4 by 9 tiles and y of the 5 by 5 tiles.

Then, by considering the area, $36x + 25y = 11200$.

Note that to make the equation work, x must be a multiple of 25 because all other terms are multiples of 25.

Also, note that with only two types of tile, to use the fewest tiles in total, it means using as many of the larger tiles as possible.

The largest multiple of 25 x can be is 300 and that would leave y being 16.

This can be achieved with the following method.

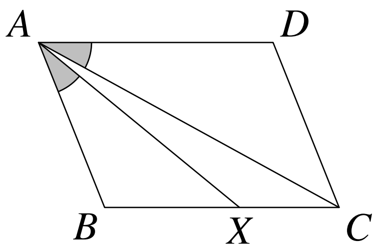
Create a rectangle 135 long by 80 wide using a grid of 15 by 20 of the 9 by 4 tiles.

Create a rectangle 5 long by 80 wide using a grid of 1 by 16 of the 5 by 5 tiles.

Join these two together to form a rectangle 140 long by 80 wide.

Therefore the smallest number of tiles required to tile the floor is 316.

3. A parallelogram $ABCD$ has a point X on BC such that $\angle BAX = \angle CAD$ and $AX : AC = 5 : 7$. Find the ratio of $CX : XB$.



SOLUTION

$\angle ABX = \angle ADC$ because opposite angles in parallelograms are equal.

Therefore $\triangle ABX \sim \triangle ADC$ because they have two, and therefore three, angles equal.

The scale factor between the triangles is $\frac{7}{5}$.

Therefore, $AD = \frac{7}{5}AB = \frac{7}{5}DC = \frac{7}{5}(\frac{7}{5}XB) = \frac{49}{25}XB$.

Therefore, $CX = BC - XB = AD - XB = \frac{49}{25}XB - XB = \frac{24}{25}XB$.

So, $\frac{CX}{XB} = \frac{24}{25}$ and $CX : XB = 24 : 25$.

4. A keypad is a 3 by 3 grid with the digits 1 to 3 in order on the top row, 4 to 6 in order on the second row and 7 to 9 in order on the third row.
Fred has the keypad upside-down (i.e rotated through 180°) and types his four-digit PIN as if the keypad were the right way up. The number he types in turns out to be 2025 more than k times his original PIN where k is a positive integer. Prove there is only one possible PIN and find it.

Solution

Note that, if the original digit in the PIN is a, the digit typed in the rotated grid is $10 - a$.

Suppose the original PIN is $x = 1000a + 100b + 10c + d$.

The number typed in is $1000(10 - a) + 100(10 - b) + 10(10 - c) + (10 - d)$.

This simplfies to $11110 - 1000a - 100b - 10c - d = 11110 - x$.

Therefore, $11110 - x = kx + 2025$.

So, $9085 = kx + x = (k + 1)x$.

Since x is a four-digit number, $k + 1$ is at most 9. Checking the factors up to 9 of 9085 and noting that $k + 1$ is at least 2, $k + 1$ can only be 5 and x must be 1817.

(In fact, the prime factorisation of 9085 is $9085 = 5 \times 1817$, but showing this takes a lot of checking and it cannot just be stated that 1817 is prime.)

The number typed in is 9293, which is 2025 more than 7268, which is 1817 multiplied by 4, so this fits the conditions of the question.

5. Find all integer values of n, $0 \le n \le 2025$, $n \ne 3$, for which $\frac{1}{n^2-9}$ can be written as a terminating decimal.

Solution

Note that to terminate when written as a decimal, the number can be written as a fraction with a denominator would be a power of 10. In simplest terms, the denominator must be a power of 2, a power of 5 or a product of a power of 2 with a power of 5.

Factorising the denominator gives $\frac{1}{n^2-9} = \frac{1}{(n+3)(n-3)}$.

The two factors of the denominator are at least -3 and differ by 6, so they have the same parity, and cannot both have a factor of 5.

If they are both odd, they must both be either 1 or a power of 5.

Checking -1, 1, 5, 25, 125 and 625, the only pair which differs by 6 is -1 and 5, so $n = 2$ is a solution.

If they are both even, one is a multiple of 4 but the other is not.

If neither has a factor of 5, they must both be even powers of 2.

Checking -2, 2, 4, 8 and 16, the only ones which differ by 6 are -2 and 4 or 2 and 8, so $n = 1$ and $n = 5$ are solutions. Above 16 the difference between consecutive pairs is greater than 6.

If the one which is a multiple of 4 has a factor of 5, then the other number must be 2 or -2, all possibilities of which have already been considered.

If the one which is not a multiple of 4 has a factor of 5, then one is a power of 2 and the other is double a power of 5.

Checking 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 with 10, 50, 250 and 1250, the pairs which differ by 6 are 4 and 10, which gives $n = 7$, 16 and 10, which gives $n = 13$, and 256 and 250, which gives $n = 253$.

The possible values for n are 1, 2, 5, 7, 13 and 253.

Alternative

Once it is established that both $n + 3$ and $n - 3$ are powers of 2, powers of 5 or products of powers of 2 and 5, it is possible to list all those numbers and find all pairs which differ by 6. Note that they could be negative, but only as small as -3. Any multiples of 5^5 are too large.

Powers of 2 are -2, -1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512 and 1024.

Powers of 2 multiplied by 5 are 5, 10, 20, 40, 80, 160, 320, 640 and 1280.

Powers of 2 multiplied by 5^2 are 25, 50, 100, 200, 400, 800 and 1600.

Powers of 2 multiplied by 5^3 are 125, 250, 500, 1000 and 2000.

Powers of 2 multiplied by 5^4 are 625 and 1250.

From this list, the same values for n are found as above.

6. A vegetable patch is a square grid of n by n uncleared cells with $n \geq 3$ and n odd. Mo the mole is in the centre cell and digs it up to clear it.
Mo then digs up cells one at a time according to the following rule. Mo can dig up an uncleared cell that is both adjacent horizontally or vertically to a cleared cell and in the middle of a line of three uncleared cells in the vegetable patch.
After digging up M squares, Mo can dig no further. Find in terms of n the smallest M for which this can happen and prove there is no smaller value.

SOLUTION

If there is a row (or column) with no cells cleared, there must be a row (or column) with no cells cleared adjacent to a row (or column) with at least one cell cleared.

If the cleared cell in this row (or column) is not at an edge, the cell next to it in the uncleared row (or column) can be cleared.

If the cleared cell in this row (or column) is at the edge, it cannot have been cleared without the adjacent cell on the same row also being cleared. The cell in this row (or column) adjacent to this cleared cell can be cleared.

Therefore, if there are no more moves available, there must be at least one cleared in every row and column.

When there are no more moves available, the cleared cells are connected and reach all four edges.

From the centre cell, there will need to be at least $n - 1$ moves up or down a column and at least $n - 1$ moves left or right along a row to reach all four edges.

Therefore the minimum number of moves is at least $1 + n - 1 + n - 1 = 2n - 1$.

This can be achieved by the following process.

From the centre square, alternately clear the one to the left of the last one cleared, then the one below the last one cleared until reaching the cell diagonally away from the corner. Then clear both the one below and the one to the left of that cell.

From the centre square, alternately clear the one above the last one cleared, then the one to the right of the last one cleared until reaching the cell diagonally away from the corner. Then clear both the one above and the one to the right of that cell.

Therefore, $M = 2n - 1$.